**Tentamen Functionaalanalyse**
**20/11/03**

1. Let $F : L^2[0,1] \to \mathbb{C}$ be defined by

$$F(f) := \int_0^1 t^2 f(t)dt, \qquad f \in L^2[0,1].$$

   (a) Is $F$ linear? Justify the answer!

   (b) Show that $F$ is bounded. Determine $||F||$.

   (c) Let $G : L^2[0,1] \to \mathbb{C}$ be a continuous linear functional defined on $L^2[0,1]$. Does there exist some $g \in L^2[0,1]$ such that $G$ is of the form

$$G(f) = 7\int_0^1 f(t)g(t)dt, \quad f \in L^2[0,1]?$$

   Justify the answer!

2. Let $T$ be a bounded linear operator from a Banach space $\mathfrak{B}$ into itself with domain $\operatorname{dom} T \subset \mathfrak{B}$. Show that $T$ is closed if and only if $\operatorname{dom} T$ is closed.

3. Let $\mathfrak{E}$ be an infinite-dimensional Banach space. Let $x \in \mathfrak{E}$, $x \neq 0$, and let $\mathfrak{M} = \operatorname{span}\{x\}$. Let $\ell : \mathfrak{M} \to \mathbb{C}$ be defined by $\ell(\lambda x) = (4i+1)\lambda||x||$, $\lambda \in \mathbb{C}$. Does there exist some $L \in \mathfrak{E}'$ ($\mathfrak{E}'$ is the dual space of $\mathfrak{E}$) such that the restriction of $L$ to $\mathfrak{M}$ is equal to $\ell$: $L|_{\mathfrak{M}} = \ell$, and

   (a) $||L|| = 4$ ?

   (b) $||L|| = \sqrt{17}$ ?

   (c) $||L|| = 5$ ?

   Justify the answers!

4. Provide the linear space $C^1[0,1]$ with

$$||x||_a := 3||x||_\infty + 2||x'||_\infty + |x(0)|, \quad x \in C^1[0,1].$$

   (a) Show that $||\cdot||_a$ is a norm on $C^1[0,1]$.

   (b) Show that $C^1[0,1]$ with the norm $||\cdot||_a$ is a Banach space.

   (c) Let

$$||x||_1 := \max\{||x||_\infty, ||x'||_\infty\}, \quad x \in C^1[0,1].$$

   Show that the norms $||\cdot||_1$ and $||\cdot||_a$ are equivalent on $C^1[0,1]$.